Course: Physics 2

Module 1

Mechanics of pointed system – solid matter

Instructor: Dr. Nguyen Thanh Son

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Module 1: Mechanics of pointed system – solid matter

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1.1. Center of mass – Motion of center of mass

1) Center of mass center of mass

- Revision of Newton’s Laws of Motion

**Newton’s First Law of Motion:** An object at rest tends to stay at rest and an object in motion tends to stay in motion with the same speed and in the same direction unless acted upon by an unbalanced force.

**Newton’s Second Law of Motion:** The acceleration of an object is directly proportional to the net force acting on it, and inversely proportional to its mass.

**Newton’s Third Law of Motion:** For every force that an object exerts on a second object, there is a force equal in magnitude but opposite in direction exerted by the second object on the first object.

- There are several different types of mechanical systems: a particle, a system of particles, a rigid object (body), and a system of linked rigid objects. Among these, the simplest mechanical system that can move is a particle. A particle has mass but no volume: mass point. Particles are the building blocks of a rigid object. The equations for a particle will be later used to develop those for a system of particles and a rigid object.

- Translational motion is the motion by which an object shifts from one point in space to another. One example of translational motion is the motion of a bullet fired from a gun.

- A rigid object is an object that retains its overall shape, meaning that the particles that make up the rigid body remain in the same positions relative to one another. In other words, the relative locations of all particles making up the object remain constant. A wheel and rotor of a motor are common examples of rigid objects that commonly appear in questions involving rotational motion.

- An ideal *rigid object* is one that is non-deformable. All real objects, however, are deformable to some extent, but the rigid object model is very useful in many situations where the deformation is negligible.

- A *rigid body* is a particular kind of physical object that occurs frequently in the study of dynamics. Specifically, an extended body that is rigid does not deform in any way, but rather always maintains its shape. A rigorous definition of this property is that any two points fixed within the body are always the same distance apart.

- Rigid body dynamics is the first step beyond basic point particle dynamics, as a rigid body is the simplest object that nevertheless has more complicated motion than a point particle. While point particles can only undergo translational motion through three-dimensional space (i.e. they have three degrees of freedom), rigid bodies additionally have the possibility of rotational motion. It can be shown that the motion of a rigid body can be described completely by translational motion of the center of mass and rotation around an axis passing through the center of mass. In other words, the motion of a
rigid body can be described as *translational motion of CM combined with rotation of the rest of the body* about an axis through the center of mass. One can treat two parts of motion separately. In fact, the translational and rotational motion of rigid bodies can be analyzed most easily with the help of a concept called the *center of mass*. The center of mass is just the average position of the mass distribution of an object.

- According to Newton’s third law, the center of mass *stays at rest* or with uniform motion when there is no net force acting on the body.

- The center of mass (CM) is thus a special point in a rigid object with position defined by

\[
\vec{r}_{CM} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i, \quad (M = \sum_{i=1}^{N} m_i \text{ is total mass of the object}) \quad (1)
\]

where \( \vec{r} \) denotes position vector.

The summation is over all particles of the object. The center of mass is actually an imaginary point whose coordinates are the mass weighted averages of the coordinates of the particles which constitute the system or the object of interest.

- In the Cartesian coordinates, we have from (1)

\[
x_{CM} = \frac{1}{M} \sum_{i=1}^{N} m_i x_i, \quad y_{CM} = \frac{1}{M} \sum_{i=1}^{N} m_i y_i, \quad z_{CM} = \frac{1}{M} \sum_{i=1}^{N} m_i z_i \quad (1')
\]

- It turns out that the motion of an object's center of mass is nearly always far simpler than the motion of any other part of the object. By restricting our attention to the motion of the center of mass, we can therefore simplify things greatly.

- We use the center of mass because it has the following important property: **All the mass of a rigid body may be assumed to be at the center of mass of the object when considering the transitional behavior of that object under the action of external forces.**

This means that the rigid body is equivalent to an equally massive point mass placed at the mass center insofar as its translational motion under a given external force is concerned. For example, consider the projectile motion of a bullet. Its center of mass follows a smooth curve, the same curve a point object or spherically symmetric object would follow if subjected to the same external forces.
• When an object experiences a pure translational motion, all of its points move with the same velocity as the center of mass; that is in the same direction and with the same speed.

• For translation of the CM a rigid body, we have

\[
\begin{align*}
\mathbf{M} \mathbf{v}_{CM} &= \sum_{i=1}^{n} m_i \mathbf{v}_i \\
\mathbf{M} a_{CM} &= \sum_{i=1}^{n} m_i a_i = \sum_i F_i = F_{\text{ext}}
\end{align*}
\]  

(2)

(2')

where \( \mathbf{v} \) denotes velocity and \( \mathbf{a} \) acceleration.

From (2) we see that the velocity of the center of mass multiplied by the total mass of the system is equal to the total momentum of the system.

From (2') we see that the acceleration of the center of mass multiplied by the total mass of the system is equal to the net external force acting on the system.

• The importance of center of mass is that it allows us to separate the transitional and rotational motions of an object. The object will be translated as though all its mass were concentrated at its mass center.

2) Motion of center of mass

• The motion of a particle is completely known if the particle’s position in space is known at all times. The position of a particle in motion is given by the position vector \( \mathbf{r} = \mathbf{OM} \), and if the particle is moving then \( \mathbf{r} \) is a function of time \( t \).

\[ \mathbf{r} = \mathbf{r}(t) \]  

(3)

• The displacement of a particle is the difference between its initial and final position vectors, denoted as \( \Delta \mathbf{r} \) and given by

\[ \Delta \mathbf{r} = \mathbf{r}_f - \mathbf{r}_i \]  

(4)

• The direction of the displacement vector is the straight line between the starting point \( M_i \) at the time \( t_i \) and the termination point \( M_f \) at the time \( t_f \) on a particle’s path (trajectory), as shown in Figure 1.
Velocity

- We now consider the change in the position vector. The average velocity \( \mathbf{v}_{\text{ave}} \) of a particle in the time interval \( \Delta t \) is

\[
\mathbf{v}_{\text{ave}} = \frac{\Delta \mathbf{r}}{\Delta t}
\]

where \( \Delta \mathbf{r} \) is the particle’s position displacement in the time interval \( \Delta t \) (\( \Delta t > 0 \)).

- Since, like displacement, \( \mathbf{v}_{\text{ave}} \) is independent of path, the direction of \( \mathbf{v}_{\text{ave}} \) is along \( \Delta \mathbf{r} \).

- If you apply a limit as \( \Delta t \) approaches 0, you obtain an instantaneous velocity \( \mathbf{v} \) at a specific point in the path. The instantaneous velocity \( \mathbf{v} \) is thus calculated by finding the time derivative of the position vector \( \mathbf{r} \)

\[
\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}
\]

- From equation (6), we see that velocity represents the rate of change of the particle’s position vector over time.

Acceleration

- The average acceleration \( \mathbf{a}_{\text{ave}} \) of a particle in motion is given by

\[
\mathbf{a}_{\text{ave}} = \frac{\Delta \mathbf{v}}{\Delta t}
\]

where \( \Delta \mathbf{v} \) is the change of the particle’s velocity in the time interval \( \Delta t \) (\( \Delta t > 0 \)).

- If you apply a limit as \( \Delta t \) approaches 0, you obtain an instantaneous acceleration \( \mathbf{a} \) at a specific point in the path. The instantaneous acceleration is thus calculated by finding the time derivative of the velocity \( \mathbf{v} \)

\[
\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}
\]

- Acceleration represents the rate of change in velocity over time.

- Combining (6) and (8) leads to
\[ a = \frac{dv}{dt} = \frac{d^2r}{dt^2} \quad (9) \]

The instantaneous acceleration is equal to the second derivative of \( r \) with respect to \( t \).

**Motion in one dimension**

- Though displacement, velocity, and acceleration are all vector quantities, in the one-dimensional case they can all be treated as scalar quantities with positive or negative values to indicate their directions. The positive and negative values of these quantities are determined by the choice of how you align the coordinate system.

**Velocity**

- The displacement in one-dimension is generally represented in regards to a starting point \( x_i \) and the end point \( x_f \) of the particle. The time that the particle in question is at each point is as denoted \( t_i \) and \( t_f \), respectively (always assuming that \( t_f \) is later than \( t_i \), since time only proceeds one way). The change in a quantity from one point to another is generally indicated with the Greek letter delta (which looks like a triangle) in the form of

\[ \Delta x = x_f - x_i \quad \text{and} \quad \Delta t = t_f - t_i \]

- Using these notations, it is possible to determine the *average velocity* \( v_{\text{ave}} \) in the following manner:

\[ v_{\text{ave}} = \frac{(x_f - x_i)}{(t_f - t_i)} = \frac{\Delta x}{\Delta t} \quad (10) \]

- If you apply a limit as \( \Delta t \) approaches 0, you obtain an *instantaneous velocity* \( v \) at a specific point in the path. Such a limit in calculus is the derivative of \( x \) with respect to \( t \).

\[ v = \frac{dx}{dt}. \quad (11) \]

- The instantaneous velocity \( v \) can be positive, negative, or zero. The instantaneous speed of a particle is defined as the magnitude of its velocity. The instantaneous speed has no direction associated with it and hence carries no algebraic sign.
Acceleration

• It is easy to quantify changes in velocity as a function of time in exactly the same way we quantify changes in position as a function of time. When the velocity of a particle changes with time, the particle is said to be accelerating. Using the terminology introduced earlier, we see that the average acceleration \( a_{\text{ave}} \) is

\[
a_{\text{ave}} = (v_f - v_i)/(t_f - t_i) = \Delta x/\Delta t
\]  

(12)

• Again, we can apply a limit as \( \Delta t \) approaches 0 to obtain an instantaneous acceleration \( a \) at a specific point in the path. The calculus representation is the derivative of \( v \) with respect to \( t \), or \( dv/dt \).

\[
a = dv/dt
\]  

(13)

• Similarly, since \( v \) is the derivative of \( x \), the instantaneous acceleration is the second derivative of \( x \) with respect to \( t \), or \( d^2x/dt^2 \).

\[
a = dv/dt = d^2x/dt^2
\]  

(14)

Constant acceleration

• If the acceleration of a particle varies in time, its motion can be complex and difficult to analyze. However, a very common and simple type of one-dimensional motion is that in which the acceleration is constant. When this is the case, the average acceleration over any time interval equals the instantaneous acceleration at any instant within the interval, the velocity changes at the same rate throughout the motion, and the instantaneous acceleration is a constant.

\[
a = \text{const}
\]  

(15)

• In several cases, such as a motion in the Earth's gravitational field, the acceleration may be constant - in other words, the velocity changes at the same rate throughout the motion.

• If we set the start time at 0 and the end time at \( t \) (by starting a stopwatch at 0 and ending it at the time of interest). The velocity at time 0 is \( v_0 \) and at time \( t \) is \( v \). From equation (13), we obtain the following equation:
\[ a = (v - v_0)/(t - 0) \]

or

\[ v = v_0 + at \]  \hspace{1cm} (16)

This powerful expression enables us to determine a particle’s velocity at any time \( t \) if we know the particle’s initial velocity \( v_0 \) and its (constant) acceleration \( a \).

- By applying some manipulations, we get:

\[ x = x_0 + v_0 t + 0.5at^2 \]  \hspace{1cm} (17)

\[ v^2 = v_0^2 + 2a(x - x_0) \]  \hspace{1cm} (18)

\[ x - x_0 = (v_0 + v)t/2 \]  \hspace{1cm} (19)

- The above equations of motion with constant acceleration can be used to solve any kinematic problem involving motion of a particle on a straight line with constant acceleration.

**Circular motion**

- Consider a disk rotating about its axis as shown in Figure 2. The axis of rotation is through the center of the disk and perpendicular to the disk’s surface. Every particle on the disk undergoes circular motion about the origin, \( O \).

**Angular position**

- *The angular position of a particle of the disk is the angle \( \theta \) between the fixed reference line in space and the line connecting the center of rotation and the particle of interest.*

We first choose a fixed reference line. Then, consider a point \( P \) at a fixed distance \( r \) from the origin \( O \), as shown in Figure 2. The point \( P \) rotates about the origin in a circle of radius \( r \).

- The *angular displacement* of the particle, \( \Delta \theta \), is defined as the angle the particle rotates in a time interval \( \Delta t \).
For circular motion one can use polar coordinates. These coordinates are convenient to use to represent the position of \( P \) (or any other point).

\( P \) is located at \((r, \theta)\), where \( r \) is the distance from the origin to \( P \) and the angle \( \theta \) is measured counterclockwise from the reference line. \( r \) is equal to the radius of the particle’s trajectory, as shown in Figure 3.

![Figure 3: Polar coordinates for circular motion.](image)

Description:

As the particle moves, the only coordinate that changes is \( \theta \).

As the particle moves through \( \theta \), it moves through an arc length \( s \), as shown in Figure 2.

The arc length \( s \) and \( \theta \) are related:

\[ s = \theta r \quad (20) \]

When \( s = r \Rightarrow \theta = 1 \text{ radian}. \]

**Average angular velocity**

- The average angular velocity, \( \omega_{ave} \), of a rotating particle is the ratio of the angular displacement \((\Delta \theta)\) to the time interval \((\Delta t)\)

\[
\omega_{ave} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta \theta}{\Delta t} \quad (21)
\]

- The instantaneous angular velocity, \( \omega \), is defined as the limit of the average angular velocity as the time interval approaches zero.

\[
\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt} \quad (22)
\]

- The instantaneous angular velocity is thus equal to the derivative of angular position \( \theta \) with respect to time \( t \).

The SI unit of angular velocity is radian/sec \((\text{rad/s or } s^{-1})\) since radian has no dimension).

Angular velocity will be positive if \( \theta \) is increasing and it will be negative if \( \theta \) is decreasing.
Angular acceleration

- Angular acceleration of a rotating particle, $\alpha_{\text{ave}}$, is defined as the ratio of the change in the angular velocity ($\Delta\omega$) to the time interval ($\Delta t$) needed for the object to undergo the change:

$$\alpha_{\text{ave}} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t} \quad (23)$$

- The instantaneous angular acceleration, $\alpha$, is defined as the limit of the average angular acceleration as the time interval goes to 0

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \quad (24)$$

- The instantaneous angular acceleration thus equals the derivative of angular velocity $\omega$ with respect to the time $t$.

The SI unit of angular acceleration is rad/s² or s⁻² since radian has no dimension.

- When a rigid object is rotating about a fixed axis, every particle of the object rotates through the same angle and has the same angular velocity and the same angular acceleration.

- When an object experiences pure rotational motion about its center of mass, all of its points move at right angles to the radius in a plane perpendicular to the axis of rotation with a speed proportional to the distance from the axis of rotation.

- Thus points on opposite sides of the axis move in opposite directions, points on the axis do not move at all since $r = 0$ and points on the outer edge move at the maximum speed.

Uniform circular motion (with no angular acceleration)

- When the instantaneous angular acceleration is zero ($\alpha = 0$), we have a uniform circular motion.

- In this case we have a constant angular velocity or $\omega = \text{constant}$, and the motion is thus periodic.

- The periodicity of a uniform circular motion is characterized by the period $T$ or the frequency $f$, given by $T = \frac{2\pi}{\omega}$ and $f = \frac{1}{T}$.

- In fact, $T$ is the time needed for the particle to make one revolution, and $f$ is the number of periods in one second.
1.2. Laws of conservation

1) Law of conservation of linear momentum

Linear momentum of a single particle

- In classical mechanics, momentum (plural momenta; SI unit: kg·m/s, or, equivalently, N·s) is the product of the mass and velocity of a particle.

\[ p = m \cdot v \]  \hspace{1cm} (25)

where \( p \) is the momentum, \( m \) the mass and \( v \) the velocity of the particle.

- It is sometimes referred to as linear momentum to distinguish it from the related quantity of angular momentum. Linear momentum is a vector quantity, since it has a direction as well as a magnitude.
  
  (i) Because mass is a positive scalar quantity, the directions of linear momentum and velocity are the same.
  
  (ii) In physical sense, linear momentum is said to signify the "quantity of motion". It is so because a particle with higher momentum generates greater impact, when stopped.
  
  (iii) According to Newton’s second law, the first derivative of linear momentum with respect to time is equal to the net external force on the particle.

\[ \frac{dp}{dt} = F_{ext} = ma \]  \hspace{1cm} (26)

where \( F_{ext} \) is the net external force on the particle, \( m \) its mass and \( a \) its acceleration.

Linear momentum of a system of particles

- The linear momentum of a system of \( N \) particles is the vector sum of the momenta of all individual particles in the system:

\[ \mathbf{P} = \sum_{i=1}^{N} m_i \mathbf{v}_i \]  \hspace{1cm} (27)

where \( \mathbf{P} \) is the total linear momentum of the particle system, \( m_i \) and \( \mathbf{v}_i \) are the mass and the velocity vector of the \( i \)-th particle, respectively, and \( N \) is the number of particles in the system.

- It can be shown that the total linear momentum of a particle system is the product of the total mass \( M \) of the system and the velocity of the center of mass \( \mathbf{v}_{CM} \).
\[ \mathbf{P} = \sum_{i=1}^{N} m_i \mathbf{v}_i = M \mathbf{v}_{CM} \]  \hspace{1cm} (28)

where \( M = \sum_{i=1}^{N} m_i \) is the total mass of the system.

- In the center of mass frame \( \mathbf{v}_{CM} \) is zero thus the linear momentum of a system is zero.

- Just like the case for a single particle, the first derivative of the total linear momentum gives the net external force on the particle system:

\[ \mathbf{F}_{\text{ext}} = \frac{d\mathbf{P}}{dt} = \mathbf{M} \mathbf{a}_{CM} \]  \hspace{1cm} (29)

where \( \mathbf{a}_{CM} \) is the acceleration of the system’s center of mass. The left hand side represents the vector sum of all forces on individual particles of the system.

- The motion of the center of mass of the system of particles actually depend only on the net external force - even though the motion of the constituent particles depend on both "internal" and "external" forces. This is the important distinction as to the roles of external and internal forces. Internal force is not responsible for the motion of the center of mass. However, motions of the particles of the system depend on both internal and external forces.

**Conservation of linear momentum**

- The law of conservation of linear momentum is a fundamental law of nature, and it states that **the total linear momentum of a closed system of particles (which has no interactions with external agents) is constant.**

- One of the consequences of this is that the center of mass of any system of particles will always continue with the same velocity unless acted on by a force from outside the system.

- Suppose there is no external force acting on a particle system, \( \mathbf{F}_{\text{ext}} = 0 \). Then from equation (29), we have:

\[ \frac{d\mathbf{P}}{dt} = 0 \]  \hspace{1cm} (30)

and therefore

\[ \mathbf{P} = \text{constant} \]  \hspace{1cm} (31)
• This equation is the mathematical form of the law of momentum conservation, stating that

\textbf{If the net external force acting on it is zero, the total momentum of a system is always conserved.}

• The internal forces acting between particles in a system do not change the total momentum of the system. Basically this is due to Newton's third law.

• Another statement of the law is \textit{when the external force on a system of particles is zero, the linear momentum of the system can not change.}

Mathematically, \[ P_f = P_i \] (32)

\textbf{Example:} A ball of mass m, which is moving with a speed \( v_1 \) in x-direction, strikes another ball of mass \( m' = 2m \), placed at the origin of horizontal planar coordinate system. The lighter ball comes to rest after the collision, whereas the heavier ball breaks in two equal parts. One part moves along y-axis with a speed \( v_2 \). Find the direction of the motion of other part in terms of \( v_1 \) and \( v_2 \). (Ans \( \theta = \tan^{-1}(v_2/v_1) \))

• If we have only one particle in motion, equation (31) becomes

\[ p = \text{const} \] (33)

\textit{2) Law of conservation of angular momentum}

\textbf{Rotational motion}

• When a rigid object rotates about a fixed axis in a given time interval, every particle of the object rotates through the same angle and has the same angular velocity and the same angular acceleration. So \( \theta, w, \alpha \) all characterize the motion of the entire rigid object as well as the individual particles of the object.

• The object keeps returning to its original orientation, so you can find the number of revolutions made by the object in a specific time. Note that one revolution corresponds to an angle of \( 2\pi \) radians.

• The techniques used in linear motion problems can be also used here.

• Under constant angular acceleration (\( \alpha = \text{const} \)), we can describe the rotational motion of the rigid object using a set of \textit{kinematic equations}. 

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• These equations are similar to the kinematic equations for linear (translational) motion. In other words, the rotational equations have the same mathematical form as the linear equations.

• One can find analogies between those quantities:

<table>
<thead>
<tr>
<th>LINEAR</th>
<th>and</th>
<th>ANGULAR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement</td>
<td>Angular displacement</td>
<td></td>
</tr>
<tr>
<td>$x$</td>
<td>$\theta$</td>
<td></td>
</tr>
<tr>
<td>Velocity</td>
<td>Angular velocity</td>
<td></td>
</tr>
<tr>
<td>$v$</td>
<td>$\omega$</td>
<td></td>
</tr>
<tr>
<td>Acceleration</td>
<td>Angular acceleration</td>
<td></td>
</tr>
<tr>
<td>$a$</td>
<td>$\alpha$</td>
<td></td>
</tr>
</tbody>
</table>

• For constant $\alpha$, we have similar results given in Table 1.1.

<table>
<thead>
<tr>
<th>Rotational Motion About Fixed Axis</th>
<th>Linear Motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_f = \omega_i + \alpha t$</td>
<td>$v_f = v_i + at$</td>
</tr>
<tr>
<td>$\theta_f = \theta_i + \omega_i t + \frac{1}{2} \alpha t^2$</td>
<td>$x_f = x_i + v_i t + \frac{1}{2} a t^2$</td>
</tr>
<tr>
<td>$\omega_f^2 = \omega_i^2 + 2 \alpha (\theta_f - \theta_i)$</td>
<td>$v_f^2 = v_i^2 + 2 a (x_f - x_i)$</td>
</tr>
<tr>
<td>$\theta_f = \theta_i + \frac{1}{2} (\omega_i + \omega_f) t$</td>
<td>$x_f = x_i + \frac{1}{2} (v_i + v_f) t$</td>
</tr>
</tbody>
</table>

Table 1.1: Kinetic equations for rotational and linear motions under constant acceleration.

• In details we have

The linear velocity $v$ is always tangent to the circular path and is called the tangential velocity, as shown in Figure 4.

The magnitude of the linear velocity is defined as the tangential speed

$$v = \frac{ds}{dt} = \frac{d(r\theta)}{dt} = r \frac{d\theta}{dt} = r\omega \quad (34)$$
The tangential acceleration, \( a_t \), is the derivative of the tangential velocity

\[
a_t = \frac{dv}{dt} = \frac{d(r\omega)}{dt} = r \frac{d\omega}{dt} = r\alpha
\]  

Figure 4: Illustration of rotational motion.

The tangential quantities depend on \( r \), and \( r \) is not the same for all points of the object.

All points on the rotating rigid object will have the same angular velocity \( \omega \), but not the same tangential velocity.

All points on the rotating rigid object will have the same angular acceleration \( \alpha \), but not the same tangential acceleration.

A particle traveling in a circle, even though it moves with a constant speed, will have a radial acceleration, \( a_r \), given by

\[
a_r = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = \frac{r^2\omega^2}{r} = r\omega^2
\]  

Therefore, each point on a rotating rigid object will experience a centripetal acceleration, \( a_c = a_r \), as shown in Figure 5.

The tangential component of the acceleration is due to the change in tangential speed (magnitude of velocity vector).

The centripetal component of the acceleration is due to change in direction of linear velocity vector.

Uniform circular motion has a centripetal acceleration \( a_c \) due to the changing direction of the linear velocity vector. General rotational motion also has this acceleration, but in addition has a linear (tangential) acceleration due to the changing magnitude of the linear velocity vector. The magnitude of the total acceleration of a particle of the rotating rigid object is given by
From equation (37) we see that the total acceleration can be found from the acceleration components.

Example a: In the early 20th century, the standard format for music recordings was a plastic disk that held a single song and rotated at 78 rpm (revolutions per minute). What was the angular velocity of such a disk? (Ans: 8.2 rad/s)

Example b: A bicycle slows down uniformly from \( v_i = 8.4 \) m/s to rest \( (v_f = 0) \) over a distance of 115 m. Diameter of wheel is 0.69 m \( (r = 0.34 \) m). Find:

(a) The initial angular velocity
(b) Total revolutions of the wheel before it comes to rest.
(c) The angular acceleration.
(d). The time to stop.

Solution

(a) \( v_i = r\omega_i \Rightarrow \omega_i = v_i/r = 8.4 \) m/s/0.34 m \( \Rightarrow \omega_i = 24.7 \) rad/s.

(b) \( \theta \equiv s/r = 115 \) m/0.34 m \( = 338.2 \) rad/2\(\pi\) rad/rev \( \Rightarrow \theta \equiv 53.8 \) rev.

(c) \( \alpha = (\omega_i^2 - \omega_f^2)/2\theta = (0 - 24.72\times24.72)/2(53.8\times2\pi) \) rad/s\(^2 \Rightarrow \alpha = -0.902 \) rad/s\(^2 \).

(d) \( t = (\omega_f - \omega_i)/\alpha = (0 - 24.7)/(-0.902) \) s \( \Rightarrow t = 27.384 \) s.

Angular momentum \( \mathbf{L} \).

Consider a particle of mass \( m \) rotates about a pivot point with an angular velocity \( \omega \) and a linear velocity \( \mathbf{v} \). We define the angular momentum \( \mathbf{L} \) of the particle about the point as

\[
\mathbf{L} = \mathbf{r} \times m \mathbf{v} \quad (38)
\]

where \( \mathbf{r} \) is the position vector of the particle of mass \( m \) with respect to the pivot point, and

\[ a = \sqrt{a_i^2 + a_z^2} = \sqrt{r^2 \alpha^2 + r^2 \omega^2} = r \sqrt{\alpha^2 + \omega^2} \quad (37) \]
\[ \vec{p} = m \vec{v} \] is its (linear) momentum. The direction of \( \vec{L} \) is perpendicular to both \( \vec{r} \) and \( \vec{p} \), as shown in Figure 6.

Let \( \vec{r} \) and \( \vec{p} \) lie in the x-y plane. Then \( \vec{L} = (r \sin \theta \ k) \), where \( k \) is the unit vector in the z direction and \( \theta \) the smaller angle between \( \vec{r} \) and \( \vec{p} \). As a result, in Figure 6 \( \vec{L} \) is out of the page.

If the center of the circular path coincides with the pivot point, then \( \theta = 90^\circ \) thus \( \sin \theta = 1 \), and \( \vec{L} = \vec{r} \vec{p} = rm \vec{v} = mr^2 \omega \) or

\[ \vec{L} = I \omega \]  \hspace{1cm} (39)

where \( I = mr^2 \).

- The rate of change in angular momentum of the particle is

\[ \frac{d\vec{L}}{dt} = \frac{d}{dt} \left( \vec{r} \times \vec{p} \right) = \vec{r} \frac{dp}{dt} + \vec{p} \frac{dr}{dt} \]

- The second term on the right is proportional to \( \vec{v} \times \vec{v} \) and therefore is zero. We thus have

\[ \frac{d\vec{L}}{dt} = \vec{r} \frac{dp}{dt} = \vec{r} \vec{x} \vec{F} = \vec{\tau} \]

where \( \vec{F} \) is the net external force, or

\[ \vec{\tau} = \frac{d\vec{L}}{dt} \] \hspace{1cm} (40)

where

\[ \vec{\tau} = \vec{r} \vec{x} \vec{F} \] \hspace{1cm} (41)

is the net torque acting on the particle. This equation is the rotational analog of Newton's second law (equation 26).

**Angular momentum of a rotating rigid object**

- The angular momentum of a rigid object rotating about one of its symmetry axes is the sum of the angular momenta of all its particles. It is a measure of an object's rotational motion about the axis. The
angular momentum \( \mathbf{L} \) then is the product of the object's moment of inertia \( I \) and its angular velocity vector \( \mathbf{\omega} \) about the axis.

\[
\mathbf{L} = I \mathbf{\omega}
\]  

(42)

where \( I \) is the inertia moment of the object about the axis of interest.

- The total angular momentum is thus a vector. The direction of the angular momentum of a rigid object rotating about one of its symmetry axes is the direction of the angular velocity \( \mathbf{\omega} \) (given by the right hand rule) because \( I \) is always positive.

- Using equation (41), if the net torque is constant we can show that the change \( \Delta\mathbf{L} \) in the angular momentum of an object is equal to the angular impulse \( \mathbf{\tau}\Delta t \). One can give an object an angular impulse by letting a torque act on it for a time interval \( \Delta t \).

\[
\text{angular impulse} = \text{torque} \times \text{time}
\]

\[
\Delta \mathbf{L} = \mathbf{\tau} \Delta t
\]  

(43)

- From equation (43) we see that if no external torque acts on the system or \( \mathbf{\tau} = 0 \), then \( \Delta \mathbf{L} = 0 \) or

\[
\mathbf{L} = \text{const}
\]  

(44)

In other words, if no external torque acts on a system of interacting objects, then their total angular momentum is conserved.

- This is the statement of the law of angular momentum conservation.

**Example:** A light rod 1 m in length rotates in the xy plane about a pivot through the rod's center (the axis of rotation is perpendicular to the xy plane). Two particles of mass 4 kg and 3 kg are connected to its ends. Determine the angular momentum of the system at the instant at which the speed of each particle is 5 m/s.

**Solution:** We assume that the mass and moment of inertia of the rod can be neglected. The moment of inertia of the system about the z-axis is 3 kg(0.5m)² + 4kg(0.5m)² = 1.75 kgm².

The angular velocity of the system is \( \mathbf{\omega} = (v/r) \mathbf{k} = ((5\text{m/s})/0.5\text{m}) \mathbf{k} = (10/\text{s}) \mathbf{k} \). The angular momentum of the system is \( \mathbf{L} = I \mathbf{\omega} = (17.5\text{ kgm}^2/\text{s}) \mathbf{k} \).
Torque

• Torque, \( \tau \), is a measure of the tendency of a force to rotate an object about some axis. Torque is a vector whose magnitude is given by

\[
\tau = r F \sin \phi = F d
\]  

(45)

where \( \vec{F} \) is the force of interest, \( \phi \) is the smaller angle the force \( \vec{F} \) makes with \( \vec{r} \), and \( d \) is the moment arm (or lever arm), as shown in Figure 7. From Figure 7, we see that \( d \) is actually the perpendicular distance from the axis of rotation to a line drawn along the direction of the force (the line of action).

• If a system is composed of many solid objects and the i-th object is acted upon by an external force \( \vec{F}_i \), corresponding to a torque \( \tau_i \), then the net torque on the system is the vector sum of individual torques

\[
\tau = \sum_{i=1}^{N} \tau_i
\]

Where \( N \) is the number of objects of the system.

1.3 Motion of a rigid object

• As we mentioned earlier, a rigid object is a particular kind of physical object which we encounter frequently in the study of dynamics. Specifically, a rigid object that is rigid does not deform in any way, but rather always maintains its shape. A rigorous definition of this property is that any two points fixed within the body are always the same distance apart.

1) Motion of a rigid object

• Rigid object dynamics is the first step beyond basic point particle dynamics, a rigid object has more complicated motion than a point particle. While point particles can only undergo translational motion
through three-dimensional space (i.e. they have three degrees of freedom), a rigid object can be described completely by translational motion of the center of mass and rotation around an axis through the center of mass.

Rigid object dynamics is a useful idealisation for the classical calculations that are used for most real-world engineering applications.

- Any motion of a rigid object can be considered as a combination of a translation and a rotation. A translation of a rigid object is a displacement of its center of mass. In other words, translational motion is the motion by which an object shifts from one point in space to another. One example of translational motion is the motion of a bullet fired from a gun.

A rotation of a rigid object is a change in its angular orientation. Rotational motion is very common. Spinning objects like tops, wheels, and the earth are all examples of rotational motion that we would like to understand.

- Of practical concern is a type of motion of rigid objects that involves both translation and rotation, called two-dimensional motion. In this type of motion, the axis of rotation has a fixed direction but not necessarily a fixed position in space. One of the interesting cases of two-dimensional motion is that of objects which roll on a surface without slipping, such as a rolling wheel, as shown in Figure 8.

- The red curve shows the path of a point on the rim of the object. This path is called a cycloid. The green line shows the path of the center of mass of the object.

![Figure 8](image.png)

**Figure 8**: An example of two-dimensional motions. The red curve shows the path of a point on the object’s rim (a cycloid). The green line shows the path of the object’s CM.

2) Moment of inertia
• Moment of inertia $I$ of an object about an axis is defined as

\[ I = \sum_i r_i^2 m_i \]  

(46)

where $m_i$ is the mass of the $i$-th particle and $r_i$ the distance from it to the axis.

• The moment of inertia is defined with respect to an axis of rotation. For example, the moment of inertia of a circular disk spinning about an axis through its center and perpendicular to the plane of the disk differs from the moment of inertia of a disk spinning about an axis through its center in the plane of the disk.

• The moment of inertia of an object depends on the mass of the object, and on how this mass is distributed with respect to the axis of rotation. The farther the bulk of the mass is from the axis of rotation, the greater is the rotational inertia (moment of inertia) of the object.

• The dimension of moment of inertia is $ML^2$ and its SI unit is kg.m$^2$.

• We can calculate the moment of inertia of an object more easily by assuming it is divided into many small volume elements, each of mass $\Delta m_i$. We can then rewrite the expression for $I$ in terms of $\Delta m_i$.

\[ I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm \]  

(47)

Since $\rho = m/V$ (volumetric mass density), for the small volume segment $\rho = dm/dV$ or $dm = \rho \, dV$, then equation (47) becomes

\[ I = \int r^2 \, dm = \int r^2 \rho \, dV \Rightarrow I = \int pr^2 \, dV \]  

(48)

• If $\rho$ is constant, the integral can be evaluated with known geometry, otherwise its variation with position must be known.

Example: Find the moment of inertia of a uniform thin hoop about its symmetrical axis.

Since this is a thin hoop, all mass elements are the same distance (R) from the center.

\[ I = \int r^2 \, dm = R^2 \int dm \Rightarrow I = MR^2 \]  

(49)

where M is the hoop’s mass and R is its radius.
Below are moments of inertia of some common objects about their symmetrical axes which contain their centers of mass.

**Parallel-axis theorem**
- In the previous examples, the axis of rotation coincides with the axis of symmetry of the object. For an arbitrary axis, the parallel-axis theorem often simplifies calculations. The theorem states mathematically:

\[ I = I_{CM} + MD^2 \]  \hspace{1cm} (50)

where \( I \) is the moment of inertia about any axis parallel to the axis through the center of mass of the object. \( I_{CM} \) is the moment of inertia about the axis through the center of mass (CM). \( D \) is the distance from the CM axis to the axis of interest.
Parallel-axis theorem example

- The axis of rotation goes through O as shown in Figure 9. The axis through CM is also shown. The moment of inertia about the axis through O would be:

\[ I_O = I_{CM} + MD^2 \]

**Figure 9:** An example of parallel-axis theorem.

Application: We want to find the moment of inertia \( I \) for a rod rotating around one end, as shown in the figure to the right. Moment of inertia of the rod about the axis through its CM is

\[ I_{CM} = \frac{1}{12} ML^2 \]

Here, \( D \) is equal to \( L/2 \). Therefore,

\[ I = I_{CM} + MD^2 \Rightarrow I = \frac{1}{12} ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3} ML^2 \]

3) Kinetic energy of a rigid object

a) Rotational kinetic energy

Work in rotational motion

- The work done by the force \( \vec{F} \) acting on the object as the object rotates through an infinitesimal distance \( ds = r \, d\theta \), as shown in Figure 10, is \( dW = \vec{F} \cdot ds = (F \sin \phi) \, r \, d\theta = \tau \, d\theta \).

- The radial component, \( F \cos \phi \), of \( \vec{F} \) does no work because it is perpendicular to the displacement \( ds \) (see Figure 10).

**Figure 10:** Illustration of work in rotational motion.
• We can show that \( dW = \tau \, d\theta = I \omega d\omega \) (51)

**Work-kinetic energy theorem in rotational motion**

• By integrating both sides of equation (51), we obtain

\[
\Delta W = \sum dW = \int_{t_0}^{t_f} I \omega d\omega = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2 \tag{52}
\]

assuming \( I \) is a constant.

• Equation (52) is the mathematical form of the work-kinetic energy theorem for rotational motion that states that *the work done by the net external force acting on a symmetrical rigid object that rotates about a fixed axis equals the change in the object’s rotational kinetic energy.*

• In fact, we can show that the rotational kinetic energy of a rotating object is given by (pages 299 and 300 Hallidays’ textbook)

\[
\text{KE}_{\text{rotation}} = \frac{1}{2} I \omega^2 \tag{53}
\]

• From equation (53) we see that the rotational kinetic energy \( \text{KE}_{\text{rotation}} \) is similar to the translational kinetic energy \( \text{KE}_{\text{translation}} = \frac{1}{2} mv^2 \) associated with linear motion.

**b) Kinetic energy of a rigid object**

• In describing motion of rolling objects, it must be kept in mind that kinetic energy is divided between translational (linear) kinetic energy \( \text{KE}_{\text{translation}} \) and rotational kinetic energy \( \text{KE}_{\text{rotation}} \). Another key is that for rolling without slipping, the linear velocity of the center of mass is equal to the angular velocity times the radius \( (v_{cm} = r\omega) \).

• *If an object is rolling without slipping, then its kinetic energy can be*

![Figure 11: Illustration of a rolling without slipping motion and the associated kinetic energy.](image-url)
expressed as the sum of the translational kinetic energy of its center of mass and the rotational kinetic energy about the axis through the object’s center of mass, as shown in Figure 11.

\[ I_{cm} \text{ is the moment of inertia about the axis through the center of mass. If it is known about some other axis, then the parallel axis theorem may be used to obtain the needed moment of inertia. } v_{cm} \text{ is the linear velocity of the object’s center of mass.} \]

**Total kinetic energy of a rolling object**

- The total kinetic energy of a rolling object is the sum of the translational kinetic energy of its center of mass and the rotational kinetic energy about the axis through its center of mass.

  \[ KE_{total} = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M v_{CM}^2 \]  
  
  \[ \text{(54)} \]

- An example of a rolling object is given by Figure 11.

**General work-kinetic energy theorem**

- It states that the work done by the net external force on an object is the change in its total kinetic energy.

  \[ \Delta W = KE_{total/f} - KE_{total/i} \]  
  
  \[ \text{(55)} \]

- At this point, it is useful to find the similarity between rotational motion and linear motion. The results are shown below:

<table>
<thead>
<tr>
<th>Rotational Motion vs Linear Motion</th>
<th>Rotational Motion</th>
<th>Linear Motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta ) (rad)</td>
<td>( \theta ) (rad)</td>
<td>( x ) (m)</td>
</tr>
<tr>
<td>( \omega ) (rad/s)</td>
<td>( \omega ) (rad/s)</td>
<td>( v ) (m/s)</td>
</tr>
<tr>
<td>( \alpha ) (rad/s²)</td>
<td>( \alpha ) (rad/s²)</td>
<td>( a ) (m/s²)</td>
</tr>
<tr>
<td>( \tau = l\alpha )</td>
<td>( \tau = l\alpha )</td>
<td>( F = ma )</td>
</tr>
<tr>
<td>( KE = (1/2)l\omega^2 )</td>
<td>( KE = (1/2)l\omega^2 )</td>
<td>( KE = (1/2)mv^2 )</td>
</tr>
<tr>
<td>( L = l\omega )</td>
<td>( p = mv )</td>
<td></td>
</tr>
<tr>
<td>( l )</td>
<td>( m )</td>
<td></td>
</tr>
</tbody>
</table>

1.4. Motion of objects with changing mass
• Up to this point we have viewed only situations in which the mass of the system remains constant. Newton himself recognized that situations in which the mass is not constant were still covered by his formulation of the second law of motion even though such situations are less common. These cases always demand that we know how to handle differential equations.

• One example where the mass does change is the case of rockets. Rockets are very useful for space travel because they carry their fuel and oxygen supply with them. In fact, most of the mass of the rocket on the ground is in the form of fuel and oxidizer. In space, the burning fuel is ejected from the rear of the rocket. This action produces a reaction force on the rocket body which propels the rocket forward. There is actually no need for any air to ‘push against’ for the rocket to work. Newton's third law assures us that ejection of an object from the system must propel the system in the opposite direction. The propulsive force is referred to as the thrust of the rocket.

• Regarding the relationship between rocket velocity and exhaust velocity: The force driving a rocket forward is an example of Newton's third law of motion. The gas goes in one direction and the rocket in the opposite direction.

**Principle of linear impulse and momentum**

• In section (1.2) we saw that Newton’s second law may be expressed in the form

\[ F = \frac{dp}{dt} \]  

(56)

• Multiplying both sides of equation (56) by dt and then integrating from a time \( t_1 \) to a time \( t_2 \), we have

\[ dp = F dt \Rightarrow \Delta p = p_2 - p_1 = \int_{t_1}^{t_2} F dt \]

or

\[ p_2 = p_1 + \int_{t_1}^{t_2} F dt \]  

(57)

• The integral in the right hand side of equation (57) is a vector known as the linear impulse, or simply the impulse, of the net force.

*Figure 12: Applying the principle of linear impulse and momentum to rocket’s motion.*
Equation (57) is the mathematical form of the principle of linear impulse and momentum that states
when a particle is acted upon by a net force $\vec{F}$ during a given time interval, the final linear
momentum $\vec{p}_2$ of the particle may be obtained by adding its initial linear momentum $\vec{p}_1$ and the
linear impulse of the net force $\vec{F}$ during that time interval.

Let us now see how we can apply the principle of impulse and momentum to rocket mechanics.
Consider a rocket of initial mass $M$ which is launched vertically at time $t = 0$. The fuel is consumed at a
constant rate $q$ ($q = -dM/dt$, positive) and is expelled at a constant speed $V_e$ relative to the rocket. At
time $t$, the mass of the rocket shell and remaining fuel is $M - qt$, and the velocity is $v$ in a fixed reference
frame. During the time interval $\Delta t$, a mass of fuel $q\Delta t$ is expelled.

We neglect the effect of air resistance. Denoting $u$ the absolute velocity (in that fixed reference frame)
of the expelled fuel and applying the principle of impulse and momentum between time $t$ and time $t+\Delta t$, we have

$$(M – qt – q\Delta t)(v + \Delta v) + q\Delta tu = (M – qt)v – g(M – qt)\Delta t$$

where $g$ is the Earth’s gravitational acceleration, as shown in Figure 12.

We divide both sides of equation (58) by $\Delta t$ and replace $u-(v+\Delta v)$ with $V_e$, the velocity of the expelled
mass relative to the rocket. As $\Delta t$ approaches zero, we obtain

$$-g(M – qt) = (M – qt)(dv/dt) -qV_e$$

Separating variables and integrating from $t = 0$, $v = v_0$ to $t = t$, $v = v$, we obtain

$$\int_{v_0}^{v} dv = \int_{0}^{t} \left(\frac{qV_e}{M – qt} – g\right) dt$$

which can be transformed into

$$v – v_0 = V_e\ln\{M/(M-qt)\} – gt$$

The term $-gt$ in equation (60) is the result of Earth's gravity pulling on the rocket. For a rocket drifting
in space, $-gt$ is not applicable and can be omitted. Equation (60) becomes
\[ v - v_0 = V_e \ln \left\{ \frac{M}{M - qt} \right\} \]  \hspace{1cm} (61)

• Furthermore, it is more appropriate to express the result in terms of velocity change, \( \Delta v \). Equation (61) can thus be written as

\[ \Delta v = V_e \ln \left\{ \frac{M}{M - qt} \right\} \]  \hspace{1cm} (62)

where \( \Delta v \) is the change in velocity in that time interval

\[ \Delta v = v - v_0 \]  \hspace{1cm} (63)

• Note that \( M = m_0 \) represents the initial mass of the rocket and \( M - qt = m \) the rocket’s mass at the final time. Therefore, equation (63) is often written as

\[ \Delta v = V_e \ln \left( \frac{m_0}{m} \right) \]  \hspace{1cm} (64)

where \( m_0/m \) is called the mass ratio. Equation (64) is also known as Tsiolkovsky's rocket equation, named after Russian rocket pioneer Konstantin E. Tsiolkovsky (1857-1935) who first derived it.

• From equation (64) we see that the increase in speed is proportional to the exhaust speed \( V_e \) and the natural logarithm of the mass ratio \( m_0/m \). To reach a high speed, the exhaust speed must be high, and the initial mass of the rocket must include the mass of a large quantity of fuel.

• In practical application, the variable \( V_e \) is usually replaced by the effective exhaust gas velocity, \( C \). Equation (64) then becomes

\[ \Delta v = C \ln \left( \frac{m_0}{m} \right) \]  \hspace{1cm} (65)

• Alternatively, we can write

\[ m = m_0 \exp\left(-\frac{\Delta v}{C}\right) \]  \hspace{1cm} (66)

• For many spacecraft maneuvers it is necessary to calculate the duration of an engine burn required to achieve a specific change in velocity. Using \( m = m_0 - qt \) and equation (66), we obtain

\[ t = \left( \frac{m_0}{q} \right) \left\{ 1 - \exp\left(-\frac{\Delta v}{C}\right) \right\} \]  \hspace{1cm} (67)
If the rocket is in outer space, far from the Earth, then there are no drag forces or gravity so there are no external forces on the rocket or the gases it expels. Therefore, if we consider the rocket and the exhaust gas to be a system, then the momentum of this system is constant with time. In this situation, the total thrust $F_{\text{thrust}}$ in this case can be shown to be written as

$$F_{\text{thrust}} = qV_e + (P_e - P_a)A_e \quad (68)$$

where $q$ is the rate of the ejected mass flow ($q = -dM/dt$, positive), $V_e$ is the exhaust gas ejection speed (relative to the rocket), $P_e$ is the pressure of the exhaust gas at the nozzle exit, $P_a$ is the pressure of the ambient atmosphere, and $A_e$ is the area of the nozzle exit. The product $qV_e$ is called the momentum, or velocity, thrust. The product $(P_e - P_a)A_e$, called the pressure thrust, is the result of unbalanced pressure forces at the nozzle exit.

Example: A rocket exhausts fuel with a velocity of 1500 m/s, relative to the rocket. It starts from rest in outer space with fuel comprising 80% of the total mass. Find its speed when all the fuel has been exhausted. Ans. 2414.6 m/s

Solution: Exhaust speed of the gases = 1500 m/s = $V_e$. Initial mass of rocket + fuel = $M = m_0$. Mass of fuel = 80% of $M$. Final mass of rocket when when all the fuel has been exhausted = 20%$M = 0.2M = m$.

Velocity of the rocket at any time is $v$, given by

$$v - v_0 = V_e \ln(m_0/m) \quad \text{(in outer space)}$$

here $v_0 = 0$ because the rocket start from rest

Where $m_0/m$ is the ratio of the initial mass of the rocket to its mass at the instant of time.

Substituting: $v = 1500 \times \ln (M/0.2M) = 2414.6 \text{ m/s}$
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